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Haar Wavelet Collocation Method for Solving Linear Volterra and Fredholm Integral Equations

Mohammed Abdujebar Essa[*1], Dr. Melisew Tefera[2], Dr. Seleshi Demie[3]

[1]Department of Mathematics, Haramaya University, P.O. Box 138, Dire Dawa, Ethiopia

[2]Department of Mathematics, Haramaya University, P.O. Box. 138, Dire Dawa, Ethiopia

[3]Department of Mathematics, Haramaya University, P.O. Box. 138, Dire Dawa, Ethiopia

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Abstract: **The main purpose of this paper is to obtain the numerical solution of linear Volterra and Fredholm integral equations by using Haar wavelet collocation method. Specifically, a numerical solution of the second kind of LinearVolterra and Fredholm integral equations has been discussed.This equation cannot be easily evaluated analytically. As a result,an efficient numerical technique has been applied to find the solution which is indeed an approximate solution. In this paper,The Haar wavelet collocation methodis used to transform linear Volterra and Fredholm integral equations in to a system of linear algebraic equations. The resulting systems of algebraic equations are solved by using Gaussian elimination with partial pivoting to compute the Haar coefficients. The presented method is verified by means of different problems, where theoretical results are numerically confirmed. The numerical results of six test problems, for which the exact solutionsare known,are considered to verify the accuracy and the efficiency of the proposed method. The numerical results are compared with the exact solutions and the performance of the Haar wavelet collocation methodis demonstrated by calculating the error norm and maximum absolute errors for different number collocation points. The computational cost of the proposed methods is analyzed by examples and the error analysis is done by Haar wavelet collocation method numerically. The convergence of the Haar wavelet collocation methodis ensured at higher level resolution (J).The numerical results show that the method is applicable, accurate and efficient. Most of computations are performed using MATLAB R2015asoftware.**

Keywords: **Integral equations; system of algebraic equations; Haar wavelets; collocation method.**

1. INTRODUCTION

Integral equation (IE) is an equation that includes the unknown function under the integral sign and the exponent of the unknown function inside the integral sign is one. Integral equation arises in several problems of science and technology and may be obtained directly from physical problems e.g., radiation transfer problems. Theory of integral equations is one of the most important branches of mathematical science and also used in mathematical tools and has wide applications in both pure and applied mathematics. Several physical situations can be modeled using IEs. The applications of IEs can be found in fluid dynamics, solid state physics, plasma physics, mathematical biology and chemical kinetics (Abbasbandy and Shivanian, 2011).

Integral equations appear in the mathematical formulations of a variety of modeling procedures. These include jump diffusion and option pricing, fluid dynamics, biomedical areas, chemical kinetics, ecology, control theory of financial mathematics, aerospace systems, industrial mathematics, etc. Initial and boundary value problems in integro-differential equations have several applications in the study of chemical, physical and biological science. Choi and Lui (1997) studied the integro-differential equation arising from an electrochemistry model. So, integral equations occur in many fields of mechanics and mathematical physics. They are also connected with problems in mechanical vibration, theory of analytic function, orthogonal systems.

Integral equations are used as mathematical models for many physical situations and integral equations also occur as reformulations of other mathematical problems. Recently a great deal of interest has been focused on the solution of integral equations by the wavelet methods, Beylkin et al. (1991) was the first paper in which the Haar wavelet method was used to solve integral equations. After that many researches employed this method to solve other types of integral equations. The basic idea of the Haar wavelet method is to convert the differential and integral equations into a system of algebraic equations. A wavelet is a mathematical function used to divide a given function or continuous-time signal into different scale components. Morlet and Grossmann first introduced the concept of wavelets in early 1980s and also, they used the French word ondelette, meaning "small wave". Soon it was transferred to English by translating "onde" into "wave", giving "wavelet". The study of wavelets has attained the present growth due to mathematical analysis of wavelets by Stromberg (1981), (Grossmann and Morlet) (1984) and Meyer (1989). Daubechies (1988) presented a method to construct wavelets with compact support and scale functions. A review of the basic properties of the wavelets and the decomposition and the reconstruction of functions in terms of the wavelet bases is given by Strang (1989). Many families of wavelets have been proposed in the literature. All these wavelet families can be classified as either being an orthogonal or biorthogonal family. Each orthogonal wavelet family is characterized by two functions- the mother scaling function and the mother wavelet. Among the wavelet families, which are defined by an analytical expression, special attention is given to the Haar wavelets. Some notable contributors include Morlet and Grossmann (1984) for formulation of continuous wavelet transform (CWT), Stromberg (1981) for early works on discrete wavelet transform (DWT), Meyer (1989) for multi-resolution analysis using wavelet transform. In 1910, Alfred Haar introduced the notion of wavelets. The Haar wavelet transform is one the earliest examples of what is known now as a compact, dyadic, orthonormal wavelet transform. Haar wavelets are made up of pairs of piecewise constant functions and are mathematically the simplest among all the wavelet families. A good feature of the Haar wavelets is the possibility to integrate those analytically arbitrary times. The Haar wavelets are very effective for treating singularities, since they can be interpreted as intermediate boundary conditions. Haar wavelets are easy to handle from the mathematical aspect. Haar wavelets are very effective for solving integral equations.

Many research papers are published by many authors for this purpose. For numerical solutions of linear integral equations, traditional quadrature formula methods and spline approximations are used. In the case of these methods systems of linear equations must be solved. For big matrices this requires a High number of arithmetic operations and a large storage capacity. A lot of computing time is saved if we succeed in replacing the fully populated transform matrix with a sparse matrix. Recently, Aziz and Siraj-ul-Islam (2013) presented a new algorithm for numerical solution of nonlinear Fredholm and Volterra integral equations using Haar wavelets.

Lepik (2009) solved the fractional integral equations by the Haar wavelet method. The Haar Wavelet Method is first applied to an equivalent integral equations system, where the solution is approximated by Haar wavelet function with unknown coefficients. Thus, simplified calculations are presented with necessary basic knowledge of Haar functions and their generation. Collocation method is used to evaluate the unknown coefficients and find the approximate solution. Shahsavaran (2011) used Haar wavelet with collocation method to solve Volterra integral equations with Weakly Singular Kernel. And also, Ghada (2018) applied for numerical solution of linear system of Fredholm integral equation using Haar wavelet collocation method. In these studies, the Advantage of using Haar wavelet collocation method is; -

(i) High Accuracy is obtained already for small number grid points.

(ii) The obtained solutions are mostly simpler compared with other known methods.

(iii) Unlike other numerical methods, Haar wavelet collocation method is found to be accurate, fast, flexible, convenient, low computational costs and is computational attractive. For this reason, Haar wavelet collocation method (HWCM) can be applied easily to solve linear Fredholm and Volterra integral equations. So, these studies highly motivated me to use the Haar wavelet collocation method for solving linear Fredholm and Volterra integral equations. Generally, the purpose of this paperused to Haar wavelet collocation method to solve linear Fredholm and Volterra integral equations. The goal is to find efficient numerical results that are more applicable to use in the real world.

2. HAAR WAVELET COLLOCATION METHOD FOR LINEAR VOLTERRA AND FREDHOLM INTEGRAL EQUATIONS

In this section, we discussedmore about Haar wavelet collocation method (HWCMS), linear Fredholm and Volterraintegral equations. The linear Fredholm and Volterra integral equations are given by the following general form.

$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt \; a \leq x, t \leq b$(2.0)

Where $\boldsymbol{\lambda}$ is constant, a function $f(x)$and kernels $k(x,t)$ are given function on the interval

$a \leq x, t \leq b$.$u$is unknown function which is to be determine.

We say that integral equation (2.0) it is linear if that which operations on unknown function in equation (2.0) its linear operations. If $a$and $b$are constants then we say this integral equation (2.0) is called linear fredholm integral equation and also if the upper limit (b) is variable x then integral equation (2.0) is called linear volterra integral equation.

Wavelet

The name of wavelet means a small wave. The family of wavelet is a set of mathematical functions that forms an orthonormal basis for the space $L^2(R)$of square integrable functions. Therefore, any square integrable function $g \in L^2(R)$ can be written as an infinite series whose terms are members of the wavelet family multiplied by some constants. Such a series is called wavelet representation of the function g.

Wavelet collocation method (WCM)

Recently, wavelet collocation method (WCM) got attention of many researchers to find the numerical solution of different problems. WCM is simply applicable and provides more efficient numerical solution. Let$I = [a, b]$ be a closed interval and take a mathematical model which is defined on the interval I.We will use the following formula for the interval I which is further divided into subintervals:

$$x_j = a + (b-a)\frac{j-0.5}{N}, \qquad j = 1,2,\dots,N$$

Where N is a positive integer and the points x_j, $j = 1,2,\dots,N$ are known as collocation points (CPs). In WCM the unknown function isapproximated using wavelets and then these approximations are substituted in the given equations. With the help of collocation points, the given equation can be converted into a system of algebraic equations.

Haar wavelet

Morlet (1982) first introduced the idea of wavelets as a family of functions constructed from dilation and translation of a single function called the “mother wavelet”. The family of Haar wavelet falls into the category of those wavelets which have compact support. Haar wavelet functions have been used from 1910 and were introduced by the Hungarian mathematician Alfred Haar (1910). The Haar functions are a family of switched rectangular wave forms where amplitudes can differ from one function to another. The Haar wavelet family for x ∈ [0, 1) is defined as follows (Lepik, 2007).

$$h_i(x) = \begin{cases} 1, & \text{for } x \epsilon [\tau_1, \tau_2) \\ -1, & \text{for } x \epsilon [\tau_2, \tau_3), i = 2.3, \dots, \text{(2.1)} \\ 0, & \text{elsewhere} \end{cases}$$

where

$\tau_1 = \frac{k}{m}, \tau_2 = \frac{k+0.5}{m}, \tau_3 = \frac{k+1}{m}$

In the above characterization, we have theinteger $m = 2^j$, where $j = 0, 1, 2, 3 \dots\dots J$. The

Integer$J$ denotes the maximal level of resolution of the Haar wavelet. Similarly, the range of the integer k is given as $k = 0,1,2, \dots m-1$. The integer k here acts as the translation parameter. The relation between the integers i, m and k is given by the equation $i = m + k + 1$.In the case, minimal values of $i$is $m = 1, k = 0$,m=1, we have$i = 2$. The maximal values of $i$is $i = 2M = 2^{J+1}$. Where $M = 2^J$ .it is assumed the value of $i = 1$, corresponds to the scaling function for the family of Haar wavelet over the interval in [0, 1]is defined as.

$h_1(x) = \begin{cases} 1, \text{ for } 0 \leq x \leq 1 \\ 0, \quad \text{otherwise} \end{cases}$(2.2)

Usually, the Haar wavelets are defined for the interval $t \in [0,1)$ but in general case $t \in [A, B]$, we divide the interval $[A, B]$ into m equal subintervals; each of the width $\Delta t = (B - A)/m$. In this case, the orthogonal set of Haar functions is defined in the interval $[A, B]$ by (Saha Ray, 2012).

$$h_0(t) = \begin{cases} 1, & \text{for } A \le t \le B \\ 0, & \text{elsewhere} \end{cases}, \quad (2.3)$$

and

$$h_i(t) = \begin{cases} 1, & \text{for} \tau_1(i) \le t < \tau_2(i) \\ -1, & \text{for } \tau_2(i) \le t < \tau_3(i) \\ 0, & \text{otherwhere} \end{cases} \quad (2.4)$$

where

$$\tau_1(i) = A + \left(\frac{k-1}{2^j}\right)(B - A) = A + \left(\frac{k-1}{2^j}\right) m\Delta t,$$

$$\tau_2(i) = A + \left(\frac{k - 0.5}{2^j}\right)(B - A) = A + \left(\frac{k - 0.5}{2^j}\right) m\Delta t,$$

$$\tau_3(i) = A + \left(\frac{k}{2^j}\right)(B - A) = A + \left(\frac{k}{2^j}\right) m\Delta t$$

For $i = 1,2,3, \ldots, m,\ m = 2^J$ and J is a positive integer which is called the maximum level of resolution. Here j *and* k represent the integer decomposition of the index i.

i.e, $i = k + 2^j - 1, 0 \le j < i \text{ and } 1 \le k < 2^j + 1.$

Function Approximation

Any function $y(t) \epsilon L^2([0,1])$ can be expanded into Haar wavelets by (Chen and Hsiao, 1997)

$$y(t) = c_0 h_0(t) + c_1 h_1(t) + c_2 h_2(t) + \cdots, \quad (2.5)$$

where

$$c_j = \int_0^1 y(t)\, h_j(t) dt$$

If $y(t)$ is approximated as a piecewise constant in each subinterval, the sum in Eq. (2.5) may be terminated after m terms and consequently we can write discrete version in the matrix form as

$$Y \approx \sum_{i=0}^{m-1} c_i h_i(t_l) = c_m{}^T H_m$$

where

Y and $c_m{}^T$ are the m-dimensional row vectors.

Here H is the Haar wavelet matrix of order m defined by $H = [h_0\ h_1\ h_2, \ldots,\ h_{m-1}]^T$, i.e.

$$H = \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{m-1,} \end{bmatrix} = \begin{bmatrix} h_{0,0} & h_{0,1} & \cdots & h_{0,m-1} \\ h_{1,0} & h_{1,1} & \cdots & h_{1,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{m-1,0} & h_{m-1,1} & \cdots & h_{m-1,m-1} \end{bmatrix}$$

where $h_{0,} h_{1,} \ldots, h_{m-1,}$ are discrete form of the Haar wavelet bases.

The collocation points are given by

$$t_l = A + (l - 0.5)\Delta t, l = 1,2, \ldots,$$

Haar wavelet functions satisfy the following properties

$$\int_0^1 h_i(x)h_l(x)dx = \begin{cases} 2^{-j}, & if \quad i = l = 2^j + k \\ 0, & if \quad i \neq 1 \end{cases}$$

and

$$\int_0^1 h_i(x)dx = \begin{cases} 1, & if\ i = 1 \\ 0, & if\ \ i = 2,3,\dots \end{cases}$$

Following Chen and Hsiao (Chen and Hsiao, 1997) and (Chen and Hsiao, 1999) the coefficients matrix $H_{il} = h_{il}(t_l)$ is introduced (this is a 2M x 2M matrix). A function u (t) which is defined in the interval $x \in [a, b]$ can be expanded into the Haar wavelet series:

$u(x) = \sum_{i=1}^{2m} a_i\, h_i(x).$ (2.6)

whereis the wavelet coefficient. The discrete form of this equation is

$u(x_l) = \sum_{i=1}^{2m} a_i\, h_i(x_l) = \sum_{i=1}^{2m} a_i\, H_{il}$(2.8)

or a matrix presentation$u = aH$, where u and a are 2M dimensional row vector

Haar wavelet collocation method for Fredholm integral equation

The linear Fredholm integral equation is of the form

$$u(x) - \int_0^1 K(x,t)u(t)\mathrm{d}t = f(x)\ , \qquad x \in [0,1] \tag{2.9}$$

Where the kernel k and the right-hand function f are prescribed and $u$is unknown functions. If we use the Haar Wavelet Collocation Method (HWCM) by approximating $u(x)$ as the Haar wavelet series

$$u(x) = \sum_{i=1}^{N} a_i h_i(x) \tag{3.0}$$

where the expansion coefficients $a_i$are unknowns and are determined as follows:

$$\sum_{i=1}^{N} a_i h_i(x) - \int_0^1 K(x,t) \sum_{i=1}^{N} a_i h_i(t)\, \mathrm{d}t = f(x)$$

or

$$\sum_{i=1}^{N} a_i h_i(x) - \sum_{i=1}^{N} a_i\, G_i(x) = f(x)$$

where

$$G_i(x) = \int_0^1 K(x,t) h_i(t)\mathrm{d}t \tag{3.1}$$

By evaluating the above equations at discrete locations, also known as, collocation points

$$x_j = \frac{j - 0.5}{N}$$

Where$N = 2^{J+1}$ and J is the maximum level of resolution for Haar wavelets (HWs). So that

$$\sum_{i=1}^{N} a_i h_i(x_j) - \sum_{i=1}^{N} a_i\, G_i(x_j) = f(x_j)$$

Let's we claim that $h_i(x_j)$ and $G_i(x_j)$ are elements of the coefficient $N \times N$ matrix H and operational $N \times N$ matrix G, respectively, such that $h_i(x_j) = H_{ij}$ and$G_i(x_j) = G_{ij}$. In addition, we denote $f(x_j)$ as components of a column vector $\vec{F}$ such that $f(x_j) = F_j$

$$\sum_{i=1}^{N} a_i H_{ij} - \sum_{i=1}^{N} a_i G_{ij} = F_j$$

In matrix notation, we can rewrite it as

$$\vec{a}^T(H - G) = \vec{F}^T (3.2)$$

whose transpose eq. (3.2), simply becomes

$$(H - G)^T \vec{a} = \vec{F} \qquad (3.3)$$

Where, $\vec{a}$ is the column vector of expansion coefficients: $\vec{a} = (a_1 a_2 \cdots a_N)^T$.The system eq. (3.3) means simply, we write in this form

$$A\vec{x} = \vec{b}$$

Where

$$A = (H - G)^T, \vec{x} = \vec{a} \text{and} \overrightarrow{b} = \vec{F}.$$

$A$is the coefficient matrix and b is the right hand column vector.

From the above system of equation (3.3)is to find the Haar coefficient $\vec{a}$ we use the following algorithm of Gaussian elimination with partial pivoting.

n = size (A, 1) we getting n

A = [A, b] this produces the augmented matrix

$Tolerance = 1 \times 10^{-12}$

Step-1 elimination process starts

For $i = 1$ to $n - 1$

$p = i$

Step-2 comparison to select the pivot

For $j = i + 1$ to n

If $|A(j, i)| > |A(i, i)|$

$U = A(i, :)$

$A(i, :) = A(j, :)$ This shows $i - th$row is equal with $j - th$row

$A(j, :) = U$

Step-3 checking for nullity of the pivots

While $|A(p, i)| < tolerance$ and $p \le n$

$p = p + 1$

if$p = n + 1$ then gives "no unique solution" else if p

$T = A(i, :)$

$A(i, :) = A(p, :)$this shows $i - th$row is equal with $p - th$row

$A(p, :) = T$

For $j = i + 1$ to n

$m = A(j,i)/A(i,i)$

For $k = i + 1$ to $n + 1$

$A(j,k) = A(j,k) - m \times A(i,k)$

Step-4 checking for non-zero of last entry

If $A(n,n) = 0$ then gives "no unique solution"

Step-5 using backward substitution

$x(n) = A(n, n+1)/A(n,n)$

For $i = n - 1$ to -1 to 1

$s = 0$

For $j = i + 1$ to n

$s = s + A(i,j) \times x(j)$

$x(i) = (A(i, n+1) - s)/A(i,i)$

Let's express the integral functions $G_i(x)$ upon substituting the expressions of Haar Wavelets. For $i = 1$

$$G_1(x) = \begin{cases} \int_0^1 K(x,t)\mathrm{d}t, & \text{for } 0 \le x \le 1, \\ 0, & \text{elsewhere} \end{cases}$$

and for $i \ge 2$

$$G_i(x) = \begin{cases} \int_{\tau_1}^{\tau_2} K(x,t)\mathrm{d}t - \int_{\tau_2}^{\tau_3} K(x,t)\mathrm{d}t, & \text{for } 0 \le x \le 1, \\ 0, & \text{elsewhere} \end{cases}$$

After we get integral functions $G_i(x)$ then we solve the systems of matrix of eq. (3.3) by using Gaussian elimination with partial pivoting and we get Haar coefficients $\vec{a}$. Once we get the Haar coefficients of $\vec{a}$ we can easily calculate the approximate solution of $u(x)$ at the collocation points. That is,

$$u(x_j) = \sum_{i=1}^{N} a_i h_i(x_j), \qquad j = 1, 2, \cdots N$$

which can be rewritten as

$$u(x_j) = \sum_{i=1}^{N} a_i H_{ij}, \qquad j = 1, 2, \cdots N$$

In matrix notation, it becomes

$$\vec{u}^T = \vec{a}^T H \tag{3.4}$$

By taking the transpose of the above eq. (3.4), we obtain

$$\vec{u} = H^T \vec{a} \tag{3.5}$$

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$\vec{u} = (u_1 u_2 \cdots u_N)^T = \left(u(x_1), u(x_2), \cdots u(x_N)\right)^T$. Here $x_1, x_2, \cdots$ are the collocation points between 0 and 1.

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where the expansion coefficients $a_i$are unknowns and are determined as follows:

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or

$$\sum_{i=1}^{N} a_i h_i(x) - \sum_{i=1}^{N} a_i\, G_i(x) = f(x)$$

where

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By evaluating the above equations at discrete locations, also known as, collocation points

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If we claim that $h_i(x_j)$ and $G_i(x_j)$ are elements of the coefficient $N \times N$ matrix H and operational $N \times N$ matrix G, respectively, such that $h_i(x_j) = H_{ij}$ and$G_i(x_j) = G_{ij}$. In addition, we denote $f(x_j)$ as components if a column vector $\vec{F}$ such that $f(x_j) = F_j$

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While $|A(p, i)| < tolerance$ and $p \leq n$

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if$p = n + 1$ then gives "no unique solution" else if $p = i$

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For $j = i + 1$ to n

$m = A(j, i)/A(i, i)$

For $k = i + 1$ to $n + 1$

$A(j, k) = A(j, k) - m \times A(i, k)$

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and for $i \ge 2$

$$G_i(x) = \begin{cases} 0, & \text{for } 0 \le x \le \tau_1, \\ \int_{\tau_1}^{x} K(x,t)\mathrm{d}t, & \text{for} \tau_1 \le x \le \tau_2, \\ \int_{\tau_1}^{\tau_2} K(x,t)\mathrm{d}t - \int_{\tau_2}^{x} K(x,t)\mathrm{d}t, & \text{for} \tau_2 \le x \le \tau_3, \\ \int_{\tau_1}^{\tau_2} K(x,t)\mathrm{d}t - \int_{\tau_2}^{\tau_3} K(x,t)\mathrm{d}t, & \text{for} \tau_3 \le x \le 1. \end{cases}$$

After we get integral functions $G_i(x)$ then we solve the systems of matrix of eq. (4.0) by using Gaussian elimination with partial pivoting and we get Haar coefficients$\vec{a}$. Once we get the Haar coefficients of $\vec{a}$ we can easily calculate the approximate solution of $u(x)$ at the collocation points. That is,

$$u(x_j) = \sum_{i=1}^{N} a_i h_i(x_j), \qquad j = 1, 2, \cdots N$$

which can be rewritten as

$$u(x_j) = \sum_{i=1}^{N} a_i H_{ij}, \qquad j = 1, 2, \cdots N$$

In matrix notation, it becomes

$$\vec{u}^T = \vec{a}^T H \tag{4.1}$$

By taking the transpose of the eq. (4.1), we obtain

$$\vec{u} = H^T \vec{a} \tag{4.2}$$

where $\vec{u} = (u_1 u_2 \cdots u_N)^T = \left(u(x_1), u(x_2), \cdots u(x_N)\right)^T$. Here $x_1, x_2, \cdots$ are the collocation points between 0 and 1.

3. NUMERICAL RESULT AND DISCUSSION

In this section, we numerically solvedifferent test Examples by using Haar wavelet collocation method.We have implemented Haar wavelet collocation method (HWCM) to solve six test Examples, from them three-test Examples for linear Fredholm integral equations and also threetest Examples for Volterra integral equations. Error functions are presented to verify the accuracy and efficiency of the following numerical results.In order to show the efficiency and accuracy of Haar wavelet collocation method, we define three kinds of errors, absolute error (E), norm error (E_2)and maximum absolute error (E_j).

$$E = \left|U_e(x_j) - U_a(x_j)\right|$$

and

$$E_2 = (\sum_{j=1}^{N} |u_a(x_j) - u_e(x_j)|^2)^{\frac{1}{2}} \text{and } E_J = \max_{1 \le j \le N} \{|u_a(x_j) - u_e(x_j)|\}$$

where $u_a(x)$ is approximate solution and $u_e(x)$ is exact solution,all numerical results have been done by MATLABR2015a software.

Numerical Examples for linear Fredholm integral equations

The procedure that used to solve the Examples of linear Fredholm integral equation shows simply we put as following

Step 1 we converts theExamples of linear Fredholm integral equation into the linear system of algebraic equations by using Haar wavelet collocation method.

Step 2 by using Gaussian elimination with partial pivoting we get the Haar coefficient of the vector $\vec{a}$ from the system of algebraic equation (3.3).

Step 3 after we get the Haar coefficients we can easily obtain the approximate solution $u(x)$ of linear Fredholm integral equation from the equation (3.5).

Example 1.(Reihani and Abadi, 2007) considerthe following linear fredholm integral equation of the second kind.

$$u(x) - \int_0^1 K(x,t)u(t)\mathrm{d}t = f(x)$$

where

$$K(x,t) = -\frac{e^{2x-\frac{5}{3}t}}{3} \qquad f(x) = e^{2x+\frac{1}{3}}$$

and its exact solution is given by

$$u(x) = e^{2x}.$$

From the Example 1, we get the following numerical results.

Table 1: The maximum absolute error E_J and norm error $E_2$for the test Example 1 in different level of resolution.

J	N	E_2	E_J	Computational Time(sec)
2	8	5.0689×10^{-3}	3.2091×10^{-3}	0.157345
3	16	1.7991×10^{-3}	8.5402×10^{-4}	0.554702
4	32	6.3671×10^{-4}	2.2028×10^{-4}	2.163138
5	64	2.2517×10^{-4}	5.5938×10^{-5}	8.771016
6	128	7.9613×10^{-5}	1.4094×10^{-5}	35.58266
7	256	2.8148×10^{-5}	3.5373×10^{-6}	141.0300

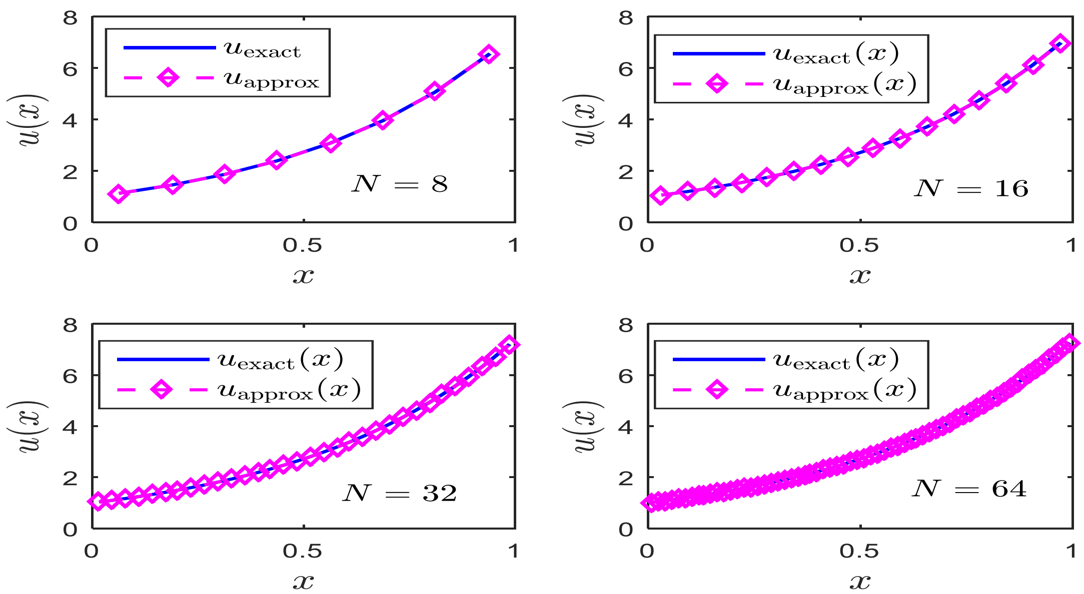

Figure 1: Comparisonof approximate and exact solution of Example 1 for N=8,N=16, N=32,N=64.

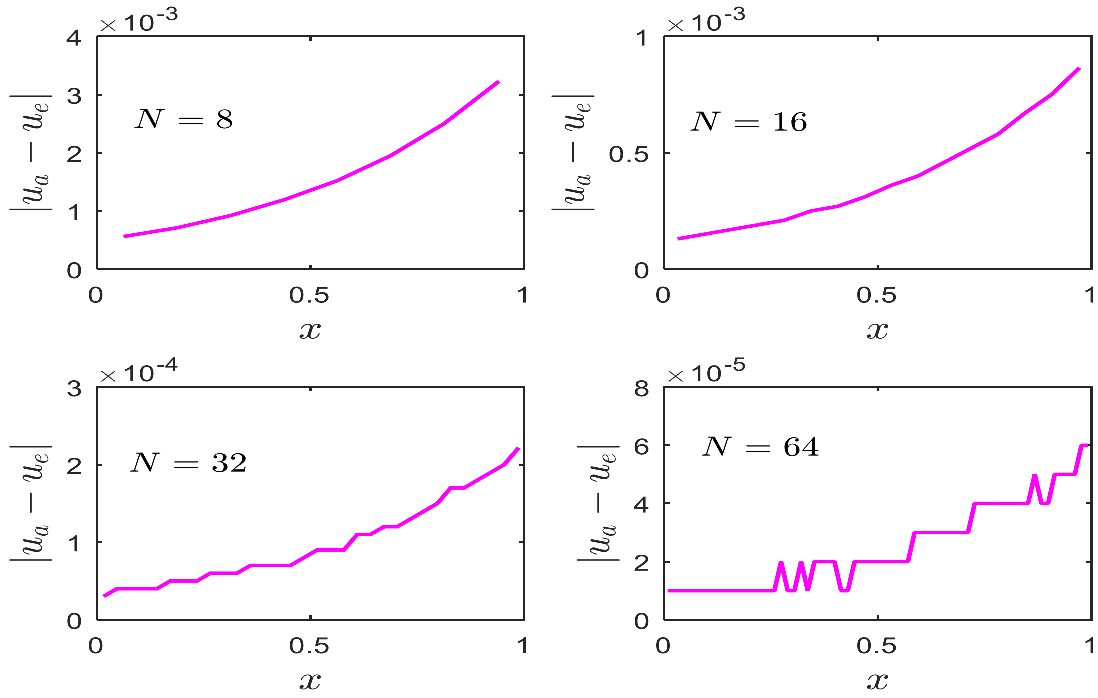

Figure 2: Absolute error of Example 1 for N=8, N=16, N=32, N=64.

Example 2: consider the following linear Fredholm integral equation of the second kind (Wazwaz, 2011):

$$u(x) - \int_0^1 K(x,t)u(t)\mathrm{d}t = f(x)$$

Where

$$K(x,t) = -2e^{x+t} \qquad f(x) = e^{x+2}$$

And its exact solution is given by

$$u(x) = e^x.$$

From the Example 2, we get the following numerical results.

Table 2: The maximum absolute error E_J and norm error E_2 for test Example 2 in different level of resolution.

J	N	E_2	E_J	Computational Time(sec)
2	8	8.5268×10^{-3}	4.3128×10^{-3}	0.161256
3	16	3.0175×10^{-3}	1.1124×10^{-3}	0.605253
4	32	1.0671×10^{-3}	2.8247×10^{-4}	2.357855
5	64	3.7730×10^{-4}	7.1170×10^{-5}	9.104379
6	128	1.3340×10^{-4}	1.7862×10^{-5}	34.50493
7	256	4.7163×10^{-5}	4.4743×10^{-6}	144.0404

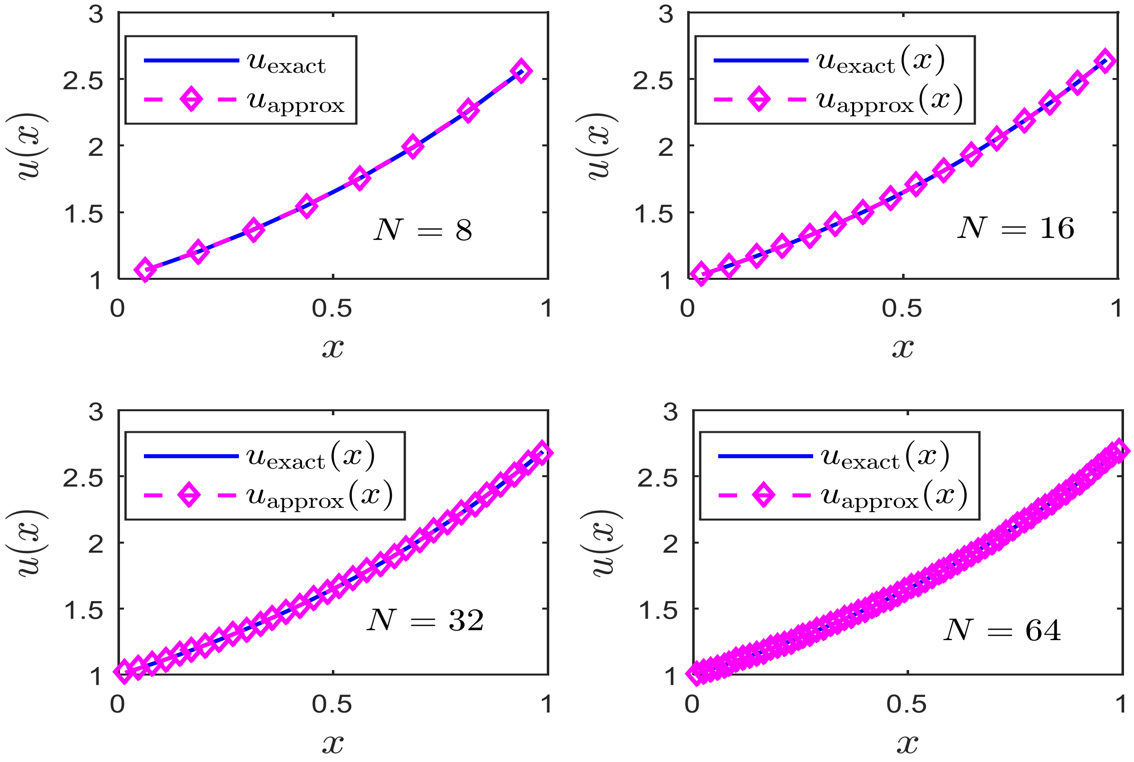

Figure 3: Comparison of approximate and exact solution of Example 2 for N=8, N=16, N=32, N=64.

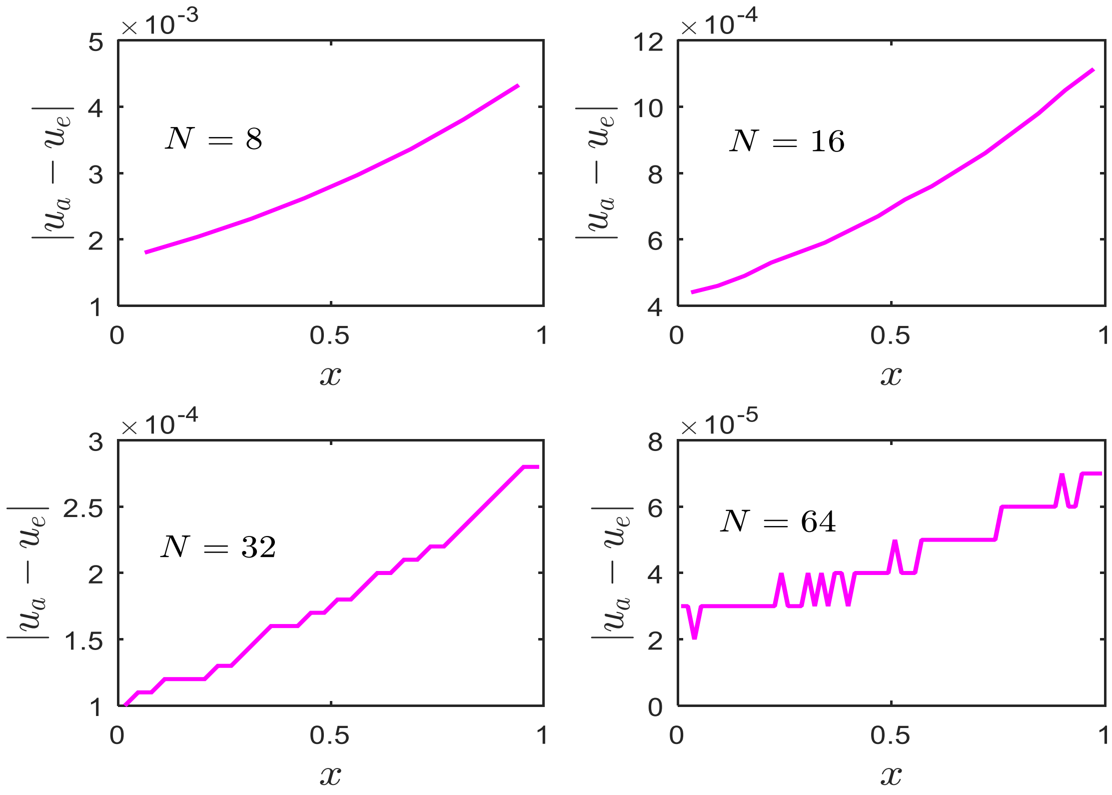

Figure 4: Absolute error of Example 2 for N=8, N=16, N=32, N=64.

Example 3: consider the following linear Fredholm integral equation of the second kind (from internet source):

$$u(x) - \int_0^1 K(x,t)u(t)\mathrm{d}t = f(x)$$

where

$$K(x,t) = -2e^{x-t} \qquad f(x) = 2xe^x$$

and its exact solution is given by

$$u(x) = e^x\left(2x - \frac{2}{3}\right)$$

From the Example 3, we get the following numerical results.

Table 3: The maximum absolute error E_J and E_2 norm error for test Example 3 in different level of resolution (J).

J	N	E_2	E_J	Computational Time(sec)
2	8	7.3026×10^{-4}	3.6935×10^{-4}	0.153789
3	16	2.5848×10^{-4}	9.5287×10^{-5}	0.556822
4	32	9.1414×10^{-5}	2.4198×10^{-5}	2.146815
5	64	3.2322×10^{-5}	6.0970×10^{-6}	9.186887
6	128	1.1428×10^{-5}	1.5302×10^{-6}	37.88264
7	256	4.0404×10^{-6}	3.8330×10^{-7}	141.6325

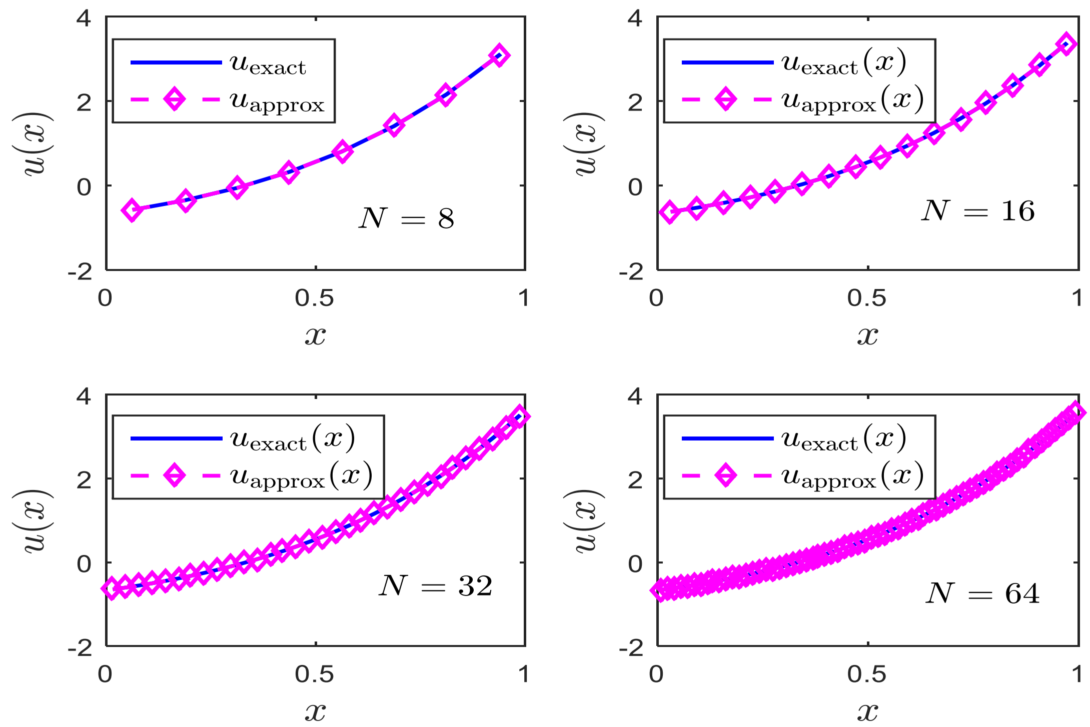

Figure 5: Comparison of approximate and exact solution of Example 3 forN=8, N=16, N=32, N=64.

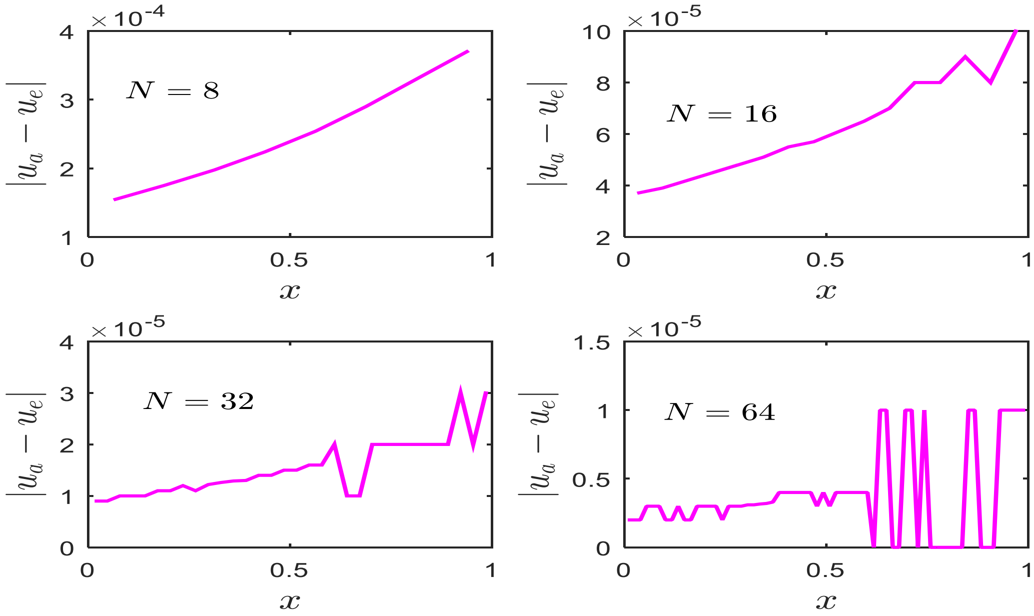

Figure 6: Absolute error of Example 3 for N=8, N=16, N=32, N=64.

As we can see the result of the above Table 1, Table 2 and Table 3, The best performance of the Haar wavelet collocation method for N=256. So, as the number of N increases, the error decreases and takes more computational time. From this we conclude that the one that have small number N are fast convergence. The accuracy of the proposed method is easily observed from the above Tables. The Table shows the maximum absolute error $(E_J)$and norm error E_2at different level of resolution of Jranging from 1 to 7 to estimate the efficiency and the accuracy of the Haar wavelet collocation method. In the above figure 1, figure 3 and figure 5, the graphs are overlappingthis shows excellent agreement of the numerical results, obtained by using Haar wavelet collocation method with the exact solution. In addition, from the Table 1, Table 2 and Table 3, as we increase the level of resolution (J) , the number of collocation points increases, the number of the dimension N increases, error decreases and computational time increases, so does the accuracy of the results of the Haar wavelet collocation method. However, the fast convergence of the method, as we can see the Table 1, Table 2 and Table 3, makes it efficient for relatively small values of N, saves the computational time.

Numerical Examples for linear Volterra integral equations

The procedure that used to solve the Examples of linear Volterra integral equation shows simply we put as following

Step 1 we converts the Examples of linear Volterra integral equation into the linear system of algebraic equations by using Haar wavelet collocation method.

Step 2 by using Gaussian elimination with partial pivoting we get the Haar coefficient of the vector $\vec{a}$ from the system of algebraic equation (4.0).

Step 3 after we get the Haar coefficients we can easily obtain the approximate solution $u(x)$of linear Volterra integral equation from the equation (4.2).

Example 4: considerthe following linear Volterra integral equation of the second kind (from the internet source):

$$u(x) - \int_0^x K(x,t)u(t)\mathrm{d}t = f(x)$$

where

$$K(x,t) = xt; \qquad f(x) = x^5 - \frac{x^8}{7}$$

and its exact solution is given by

$$u(x) = x^5.$$

From the Example 4, we get the numerical results as the following.

Table 4: The maximum absolute error E_J and norm error E_2 for test Example 4 in different level of resolution(J).

J	N	E_2	E_J	Computational Time(sec)
2	8	4.8160×10^{-3}	4.4184×10^{-3}	0.126229
3	16	1.7782×10^{-3}	1.3607×10^{-3}	0.351672
4	32	6.3551×10^{-4}	3.7663×10^{-4}	1.268755
5	64	2.2529×10^{-4}	9.9027×10^{-5}	4.781148
6	128	7.9707×10^{-5}	2.5386×10^{-5}	19.50743
7	256	2.8185×10^{-5}	6.4264×10^{-6}	74.66866

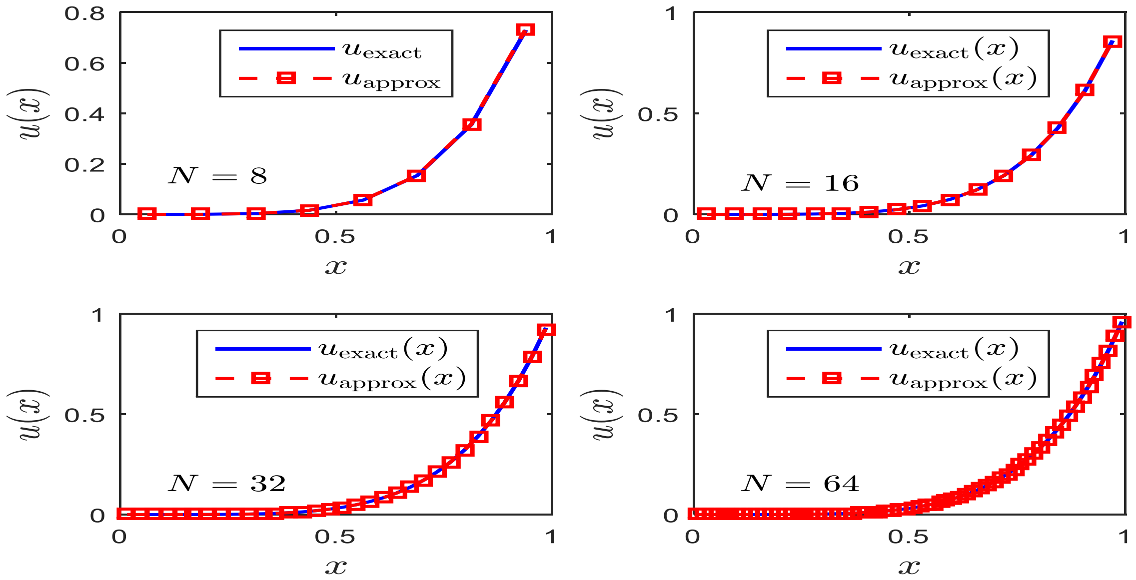

Figure 7: Comparison of approximate and exact solution of Example 4 for N=8, N=16, N=32, N=64.

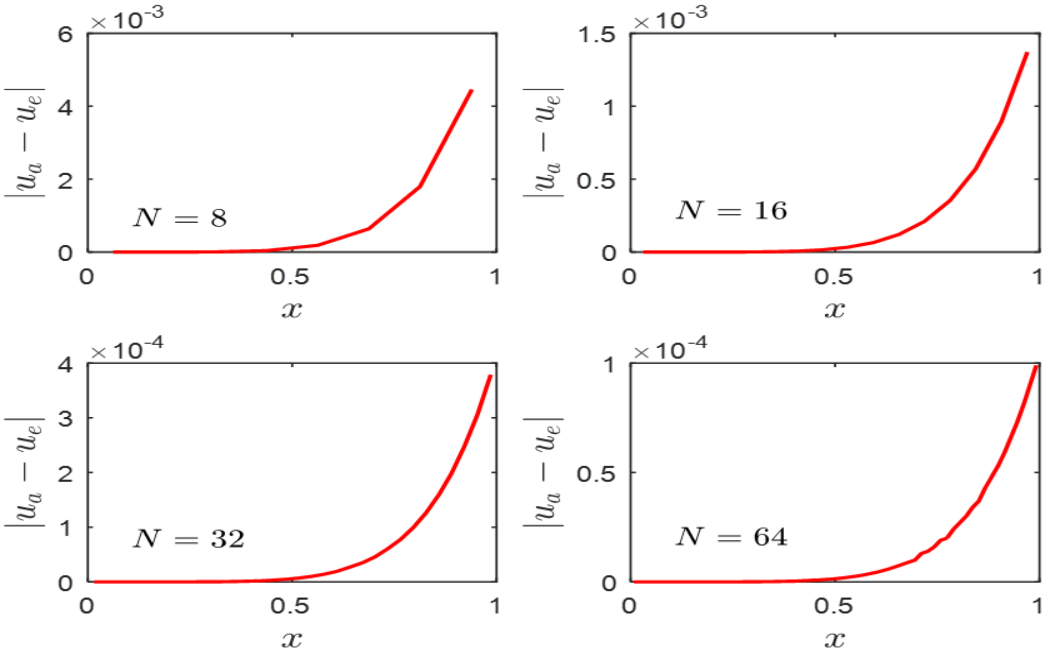

Figure 8: Absolute error of Example 4 for N=8, N=16, N=32, N=64.

Example 5: considerthe following linear Volterra integral equation of the second kind (from internet source):

$$u(x) - \int_0^x K(x,t)u(t)\mathrm{d}t = f(x)$$

where

$$K(x,t) = -\frac{t}{1+x^2}\,; \qquad f(x) = \frac{1}{1+x^2}$$

and its exact solution is given by

$$u(x) = (1+x^2)^{-3/2}$$

From the Example 5, we get the numerical results as the following.

Table 5: The maximum absolute error E_J and norm error E_2 for test Example 5 in different level of resolution(J).

J	N	E_2	E_J	Computational Time(sec)
2	8	5.4538×10^{-4}	2.7875×10^{-4}	0.132605
3	16	1.9359×10^{-4}	7.0247×10^{-5}	0.346192
4	32	6.8512×10^{-5}	1.7546×10^{-5}	1.262279
5	64	2.4229×10^{-5}	4.3921×10^{-6}	4.762492
6	128	8.5667×10^{-6}	1.0984×10^{-6}	19.05888
7	256	3.0288×10^{-6}	2.7461×10^{-7}	76.02191

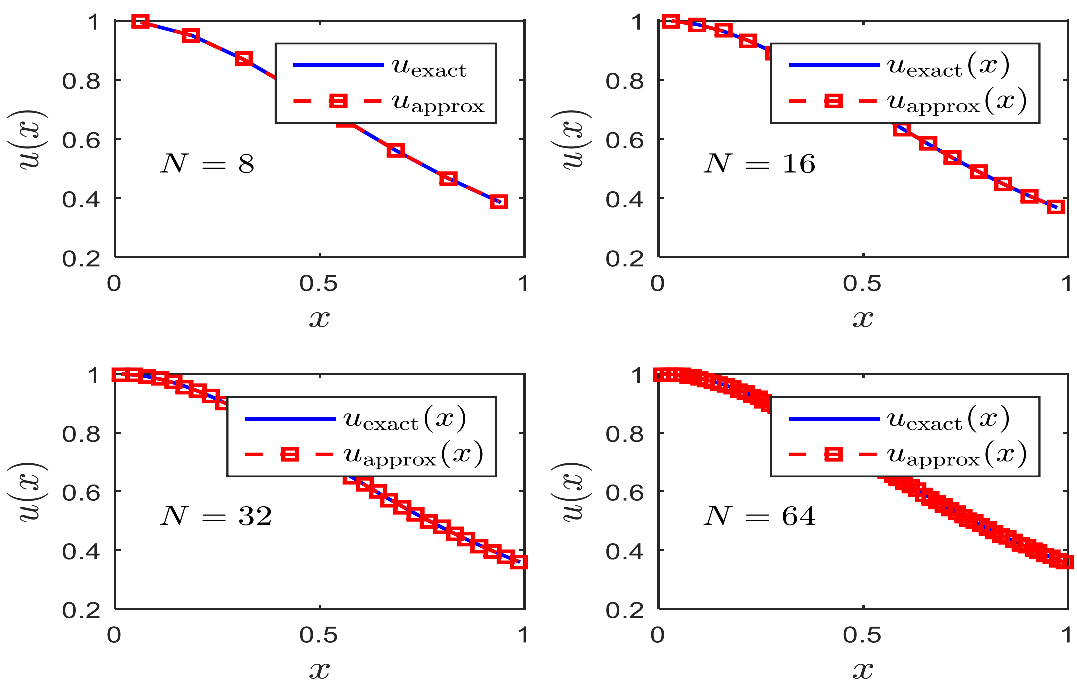

Figure 9: Comparison of approximate and exact solution of Example 5 for N=8, N=16, N=32, N=64.

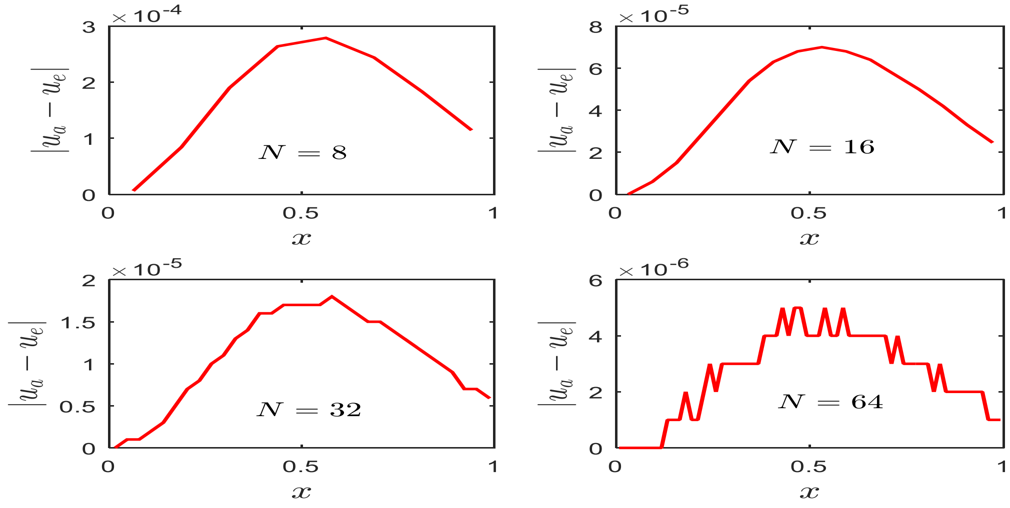

Figure 10: Absolute error of Example 5 for N=8, N=16, N=32, N=64.

Example 6: consider the following linear Volterra integral equation of the second kind (Babolian and Davari, 2005):

$$u(x) - \int_0^x K(x,t)u(t)\mathrm{d}t = f(x)$$

where

$$K(x,t) = t - x \qquad f(x) = x$$

and its exact solution is given by

$$u(x) = \sin x$$

From the Example 6, we get the numerical results as the following.

Table 6: The maximum absolute error E_J and norm error E_2 for test Example 6 in different level of resolution(J).

J	N	E_2	E_J	Computational Time (sec)
2	8	1.8223×10^{-3}	9.6613×10^{-4}	0.117262
3	16	6.4547×10^{-4}	2.4576×10^{-4}	0.345720
4	32	2.2831×10^{-4}	6.1914×10^{-5}	1.229137
5	64	8.0729×10^{-5}	1.5534×10^{-5}	4.909744
6	128	2.8543×10^{-5}	3.8904×10^{-6}	20.02375
7	256	1.0092×10^{-5}	9.7342×10^{-7}	76.53057

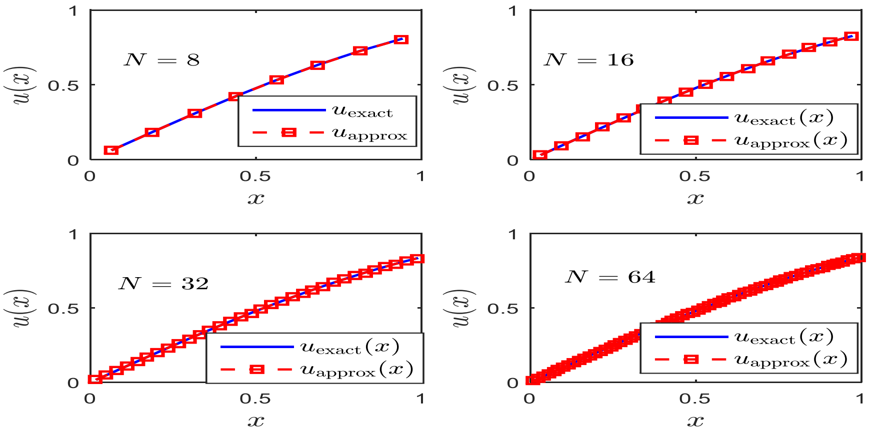

Figure 11: Comparison of approximate and exact solution of Example 6 for N=8, N=16, N=32, N=64.

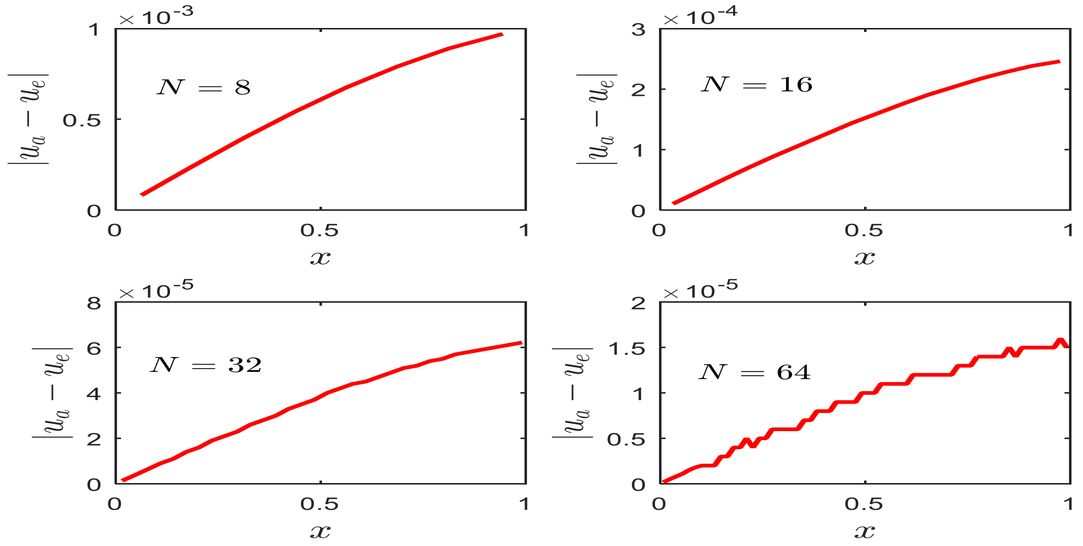

Figure 12: Absolute error of Example 6 for N=8, N=16, N=32, N=64.

From the Example 4, Example 5 and Example 6, we can be easily observed that the Haar wavelet collocation method (HWCM) can treat the problems easily, even if, when we see the Table 4,Table 5 and Table 6 and the graph of the Figure 7, Figure 9 and Figure 11, the resulting are excellent agreement with the exact solutions. Thus, as we can see the above Figure 7, Figure 9 and Figure 11, the graphs are overlapping this implies the solutions we obtained by Haar wavelet collocation method gives more accurate results as well as the results from accurate solutions.Table 4, Table 5 and Table 6 shows the maximum absolute error of the proposed method for different numbers of collocation points. From the Table 4, Table 5 and Table 6, we can see that the performances of the present method are better as the number of collocation points increases. The maximum absolute errors are decreased to order 10^{-7} for just N=256 and the numbers of collocation points which shows the better performance of the proposed in terms of accuracy, also the proposed method is getting better and better as the number of collocation points increases. The maximum absolute error (E_J) and norm error (E_2)determined the accuracy of Haar wavelet collocation method (HWCM). So, from the numerical result of Table 4, Table 5 and Table 6, the one that have small number N takes small computational time, and fast convergence. However, the one that have large number N takes more computational time, low convergence and gives more accurate results.

4. CONCLUSION

In this paper, the Haar wavelet collocation method for solution of linear integral equation is proposed. Amethod of solution which is applicable for different kind of integral equations. Fredholm and Volterra integral equations are worked out. The benefits of the Haar wavelet collocation method are sparse matrices of representation, fast transformation and possibility of implementation of fast algorithm. The numerical results obtained by the Haar wavelet collocation method are excellent agreement with the exact solutions. Maximum absolute error (E_J) and norm error (E_2) shows the better performance of the proposed method. The numerical results of the given test problem shows, as the values of the number N increases, maximal level of resolution (J) increases, error function (Maximum Absolute Error) decreases and computational cost increases and we would get more collocation points and also more accurate results. However, the one that have a small number N are given fast convergence and saves computational time. Furthermore, numerical results of all different test problems could be implemented in MATLAB R2015a software. Finally, from the result of the given test problems, we obtain the performance Haar wavelet collocation method are more efficient and accurate in the comparison to the exact solution.

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